

Synchronization of stochastically coupled cellular automata

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We analyze the dynamics of two coupled elementary cellular automata with complex spatiotemporal dynamics. The coupling mechanism, characterized by a probability p , is stochastic. From numerical simulations, we find that a sharp transition to full synchronization of the two cellular automata occurs at a critical value of p . This transition admits an approximate semianalytical description in terms of the evolution of the difference automaton, and is found both for symmetric and asymmetric coupling. It is argued that the behavior of cellular automata under coupling may inspire a classification of their individual dynamics. [S1063-651X(98)51407-9]

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The phenomenon of synchronization of coupled chaotic dynamical systems has recently inspired a considerable amount of research. The mutual engagement of two or more elements with complex individual dynamics into a fully synchronized state whose evolution is still complex has been pointed out as a relevant feature in the behavior of a wide class of natural systems [1], ranging from catalytic chemical reactions on surfaces [2] to biological systems such as neural networks [3] and insect populations [4]. In its simplest version, synchronization has been observed in systems of two identical elements whose respective individual dynamics, which are assumed to be governed by the equation $\dot{\mathbf{w}} = \mathbf{F}(\mathbf{w})$, are coupled to each other in the form

$$\dot{\mathbf{w}}_{1,2} = \mathbf{F}(\mathbf{w}_{1,2}) + \frac{\epsilon}{2}(\mathbf{w}_{2,1} - \mathbf{w}_{1,2}) = \mathbf{F}(\mathbf{w}_{1,2}) + \epsilon(\bar{\mathbf{w}} - \mathbf{w}_{1,2}). \quad (1)$$

The coupling term, weighted by the coupling intensity ϵ , describes a kind of relaxation of the state of each element to the state of the other or, equivalently, to the average state $\bar{\mathbf{w}} = (\mathbf{w}_1 + \mathbf{w}_2)/2$. For this form of coupling it has been shown that (if ϵ is above a certain critical value, related to the maximum Lyapunov exponent of the individual dynamics) both elements approach a state of full synchronization in which $\mathbf{w}_1(t) = \mathbf{w}_2(t)$ and such that the synchronized pair reproduces the dynamics of a single, uncoupled element [5].

Very recently, the possibility of synchronizing two *spatially extended* chaotic elements, whose individual dynamics is governed by the complex Ginzburg-Landau equation, has been proven [6]. In this Rapid Communication, we characterize the phenomenon of synchronization between a different kind of extended systems, namely, cellular automata. Cellular automata were initially introduced to mimic self-reproducing machines and as universal computers. After the work by Wolfram, who conjectured that the laws of nature could be given in terms of such fully discretized entities, they have been extensively applied to model physical, biological, and artificial systems [7]. Systematic study of their dynamics has shown that some cellular automata are able to

exhibit novel (and very complex) forms of behavior whose characterization, in fact, has still to be completely achieved [8]. They constitute a distinct class of dynamical systems, that is worthy of separate detailed study.

As stated above, we are here interested at analyzing synchronization of two coupled cellular automata (CA). We consider CA of the most elementary class, consisting of a linear array of N cells that adopt two possible states, $\sigma_i = 0$ or $\sigma_i = 1$, with $i = 1, \dots, N$. The individual evolution is deterministic and discrete in time, and the state of each cell at each step is a prescribed function of the previous state of the same cell and of its two nearest neighbors, $\sigma_i(t+1) = f[\sigma_{i-1}(t), \sigma_i(t), \sigma_{i+1}(t)]$. Periodic boundary conditions are assumed. Even for these simplest CA a wide variety of behaviors has been observed, depending on the form of the evolution rule f [9]. Some of them can even exhibit nonperiodic evolution in the limit $N \rightarrow \infty$, displaying spatiotemporal structures that mimic those observed in chaotic extended systems (cf., for instance, Refs. [6] and [9]). Here we are mainly interested in rules that give place to such kinds of nontrivial evolution.

As a starting point, we choose two CA with the same evolution rule f but with different random initial conditions. The state of the respective cells will be denoted by $\sigma_i^1(t)$ and $\sigma_i^2(t)$, $i = 1, \dots, N$. Due to the very character of these fully discrete systems, it is not possible to introduce a form of deterministic coupling able to be controlled with a continuous parameter such as the coupling intensity ϵ in Eq. (1). We introduce instead a form of stochastic coupling, characterized by a probability p ($0 \leq p \leq 1$), as follows. At each time step, the dynamics of the coupled CA is given by the successive application of two evolution operators to the state vector of each automaton, $\boldsymbol{\sigma}^{1,2}(t+1) = \hat{S} \circ \hat{R}(\boldsymbol{\sigma}^{1,2}(t))$. The operator \hat{R} stands for the independent evolution of each CA, according to the corresponding rule, $\hat{R}_i(\boldsymbol{\sigma}^{1,2}) = f[\sigma_{i-1}^{1,2}, \sigma_i^{1,2}, \sigma_{i+1}^{1,2}]$. The stochastic coupling operator \hat{S} compares the states σ_i^1 and σ_i^2 of the homologous cells in each CA for all i . If $\sigma_i^1 = \sigma_i^2$, both states are kept invariant. Otherwise, with probability $1-p$ they are also left unchanged, but with the complementary probability p , the states are made equal either to σ_i^1 or to σ_i^2 . This latter choice is made with probability $1/2$. Therefore,

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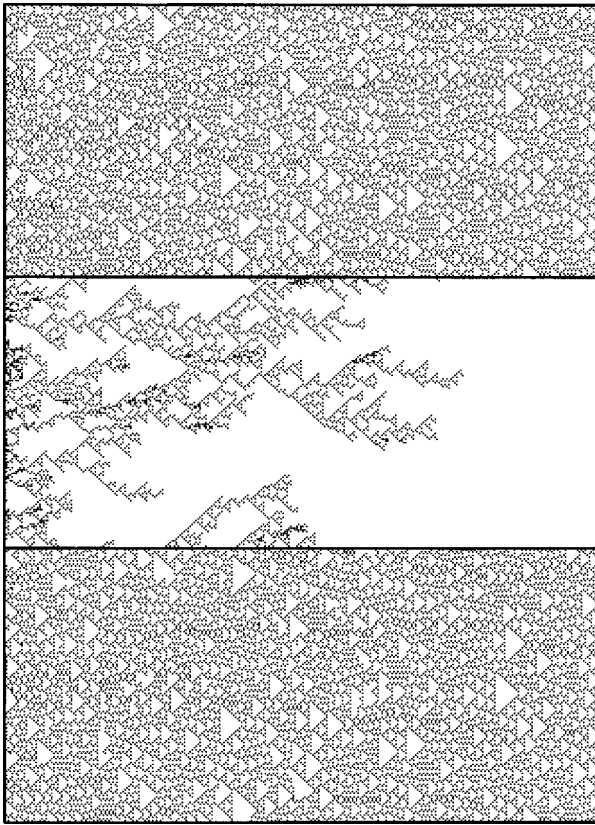


FIG. 1. Evolution of two coupled rule-18 cellular automata, with a coupling probability $p=0.2$, just above the synchronization transition. The temporal axis points rightwards. The upper and the lower plots show 300 successive states of the two automata, whereas the central plot displays the corresponding difference automaton. Each automaton has 500 cells but, for the sake of clarity, only a selected region of 150 cells is shown.

$$\hat{S}_i(\sigma^{1,2}) = \begin{cases} \sigma_i^{1,2} & \text{if } \sigma_i^1 = \sigma_i^2, \\ \sigma_i^{1,2} & \text{with probability } 1-p, \text{ if } \sigma_i^1 \neq \sigma_i^2, \\ \sigma_i^1 & \text{with probability } p/2, \text{ if } \sigma_i^1 \neq \sigma_i^2, \\ \sigma_i^2 & \text{with probability } p/2, \text{ if } \sigma_i^1 \neq \sigma_i^2. \end{cases} \quad (2)$$

This form of coupling, in which the state of homologous cells in each CA are occasionally made equal, adapts to the discrete evolution of these systems the process of mutual relaxation implied by the coupling terms of Eq. (1). On the other hand, it introduces a novel stochastic element that immediately raises the question on whether this random coupling is as efficient in producing synchronization as deterministic mechanisms.

It is clear that when the *coupling probability* p vanishes, the two CA evolve independently of each other. From different random initial conditions, and except for a few trivial evolution rules, their states will always differ to some extent. On the other hand, for $p=1$, the two CA are fully synchronized after the first evolution step. From then on, they will reproduce the evolution of a noncoupled CA but their states will be mutually identical. This suggests that some kind of transition should occur in the interval $0 < p < 1$. As we show below, in fact, a rather abrupt transition between desynchro-

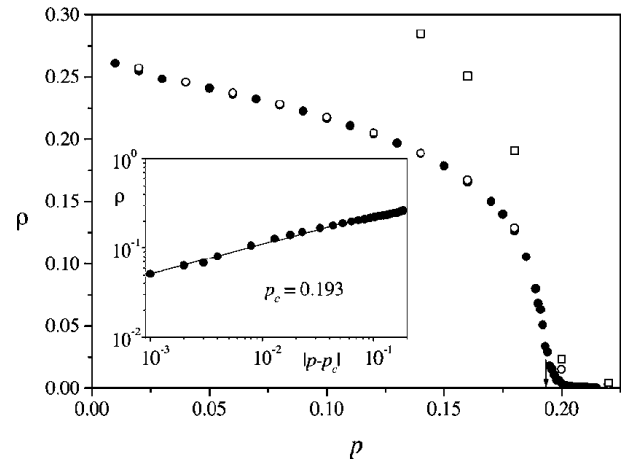


FIG. 2. Asymptotic density of the difference automaton ρ as a function of the coupling probability p . Full (empty) dots correspond to two symmetrically (asymmetrically) coupled cellular automata. Squares correspond to three coupled cellular automata. The arrow in the horizontal axis indicates the value of $p_c=0.193$. Inset: Log-log plot of ρ vs $|p-p_c|$, with $p_c=0.193$. The straight line is a least-square linear fitting, with slope $\beta=0.34$.

nization and full synchronization takes place for nontrivial evolution rules at a critical value of the coupling probability.

To study the behavior of the two coupled CA at intermediate values of p we have performed extensive numerical simulations. We have focused our attention on the so-called rule 18 [9], for which $f[0,0,1]=f[1,0,0]=1$ and $f[\sigma_{i-1}, \sigma_i, \sigma_{i+1}]=0$ otherwise. Rule 18 gives rise to non-periodic evolution in the limit $N \rightarrow \infty$, and generates homogeneous but disordered triangular spatiotemporal patterns (Fig. 1). In our simulations, we have taken $N=600$ and $N=1000$. Each realization consists of a first stage where the two CA with random initial configurations are left to evolve without coupling during 200 time steps. During this stage (whose length, as we have tested, is generically independent of N for rule 18) transients elapse and the CA enter their asymptotic regime, characterized by typical spatiotemporal triangle-like structures (see Fig. 1). Once the CA are within this regime, coupling is switched on.

In order to study the properties of synchronization of the CA we have analyzed the evolution of the *difference automaton* (DA), in which the state of the i th cell is defined as the Boolean difference between $\sigma_i^1(t)$ and $\sigma_i^2(t)$, $\delta_i(t) = \sigma_i^1(t) \oplus \sigma_i^2(t) = |\sigma_i^1(t) - \sigma_i^2(t)|$. As a ‘‘macroscopic’’ quantity characterizing the state of the DA we have chosen its density $\rho(t)$, defined as

$$\rho(t) = \frac{1}{N} \sum_{i=1}^N \delta_i(t), \quad (3)$$

($0 \leq \rho(t) \leq 1$). For sufficiently large values of p we have found that $\lim_{t \rightarrow \infty} \rho(t) = 0$, corresponding to full synchronization of the CA, as illustrated in Fig. 1, whereas for small coupling probabilities, $\lim_{t \rightarrow \infty} \rho(t) = \rho \neq 0$. The asymptotic value ρ of the DA density seems therefore to provide a suitable *order parameter* for the synchronization transition. Figure 2 shows ρ as a function of p , averaged over 50 realizations that differ in the initial conditions of the CA and over

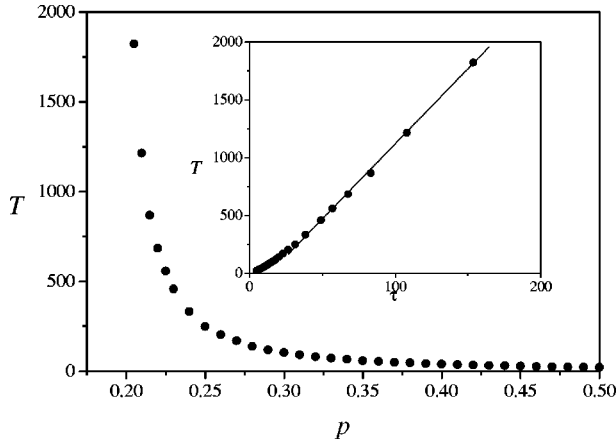


FIG. 3. Synchronization time T as a function of the coupling probability p . Inset: T as a function of the relaxation time τ . The straight line stands for the linear fit for $\tau > 30$. Its slope is about 12.8.

10^4 time steps. For small p , ρ varies rather smoothly from its value with no coupling. As the coupling probability approaches the value $p \approx 0.2$, however, a sudden decay of ρ is observed. For larger values of p a tiny exponential tail is observed and, finally, the asymptotic DA density vanishes identically. The critical-like behavior of the appearance of synchronization is therefore apparent. This suggests the existence of a critical coupling probability, $p_c \approx 0.2$, above which the evolution of the two CA is fully synchronized. Note that the fact that we do not observe a completely sharp transition in ρ as a function of p can reasonably be ascribed to the finiteness of the computation time in our numerical simulations. Occasionally, in fact, some structures in the DA are extremely persistent just above p_c .

The synchronization regime, $p > p_c$, could also be characterized by means of a single nonvanishing parameter, namely, the synchronization time. In fact, the time T that full synchronization takes to settle down from the moment when coupling is switched on depends strongly on the coupling probability. For a single realization, T is defined by the condition $\rho(t) \neq 0$ for $t < T$ but $\rho(t) = 0$ for $t \geq T$. Figure 3 shows a plot of T as a function of p , averaged over 500 realizations. We see that it diverges as the coupling probability approaches p_c . The identification of T^{-1} , or a similar function of T , as an order parameter would however be misleading, as the synchronization time stands for a dynamical feature of the transient evolution and does not characterize the synchronized state.

The form of the asymptotic DA density as a function of the coupling probability for $p < p_c$ (Fig. 2) calls for a fitting of these data with a critical-like power-law function

$$\rho = A|p - p_c|^\beta. \quad (4)$$

To perform this fitting, we have constructed a log-log plot of ρ versus $|p - p_c|$ for different values of the critical probability and looked for the value of p_c that gave the best linear correlation for the numerical data, with $\rho > 0.03$, in order to avoid the exponential tail observed for small DA densities. The slope of the least-square linear fitting gives the critical exponent β . This optimal fitting is achieved for $p_c = 0.193$

± 0.001 , and the corresponding critical exponent is $\beta = 0.34 \pm 0.01$. The log-log plot for this value of p_c is shown in the inset of Fig. 2, where the straight line stands for the linear fitting. Remarkably enough, this fitting holds practically in the whole domain of coupling probabilities. Note moreover that β is suggestively close to $1/3$.

These numerical results make it possible to build up a description of the coupled CA dynamics (and, in particular, of the synchronization transition) in terms of the evolution of a single ‘‘macroscopic’’ quantity, namely, the DA density. Let us assume that, for two uncoupled CA, the DA density evolves according to $\rho(t+1) = \phi(\rho(t))$. It is therefore clear that, in the presence of the stochastic coupling mechanism described above, the evolution of $\rho(t)$ will be governed by the equation

$$\rho(t+1) = (1-p)\phi(\rho(t)). \quad (5)$$

Now, for $p < p_c$, the nonvanishing asymptotic value of the DA density should satisfy $\rho = (1-p)\phi(\rho)$, which allows us to obtain $\phi(\rho) = \rho/[1 - p(\rho)]$, where the coupling probability p has to be seen as a function of ρ , through its connection with the stationary value of the DA density. According to our results, this connection can be well approximated by Eq. (4), with suitable values of A , p_c , and β . We can thus estimate $\phi(\rho)$ for $0 \leq \rho \leq \rho_0$, i.e., for $0 \leq p \leq p_c$, as $\phi(\rho) = \rho/[1 - p_c + (\rho/A)^{1/\beta}]$. For small values of ρ , i.e. near the synchronization transition, ϕ is linear in ρ , $\phi(\rho) \approx \rho/(1 - p_c) \approx 1.24\rho$, and has a higher-order correction of order $\rho^{1+1/\beta}$ which, in general, is nonanalytical.

According to Eq. (5), a leading linear term in $\phi(\rho)$ implies that, within the synchronization regime and for $p \approx p_c$, $\rho(t) \approx [(1-p)/(1-p_c)]^t \rho(0)$ if $\rho(0)$ is small. This predicts relaxation to $\rho = 0$ within a typical time $\tau = -1/\ln[(1-p)/(1-p_c)]$. It can be reasonably assumed that the synchronization time T measured in our simulations near the transition is proportional to τ . This implies that, for $p \approx p_c$, $T = B|p - p_c|^{-1}$, with B a constant. This formulation, therefore, predicts an integer exponent for the divergence of the synchronization time as $p \rightarrow p_c$ from above. To test this prediction, we have plotted, in the inset in Fig. 3, the synchronization time T as a function of the predicted relaxation time $\tau \sim |p - p_c|^{-1}$. As expected, they are linearly correlated for large values of τ .

In order to test the robustness of stochastic coupling between CA, we have analyzed some variants, namely, two-CA asymmetric coupling and three-CA symmetric coupling. In the first case, coupling acts in one direction only, so that one of the CA is slaved to the other, which evolves freely. Three-CA symmetric coupling is a straightforward extension of the situation considered in this Rapid Communication. In both cases, we have found that, for rule 18, a synchronization transition of the same type as described above occurs (Fig. 2). Remarkably, the critical value of p is, within our numerical precision, the same as for two symmetrically coupled CA, which seems to indicate that p_c is fully determined by some feature of the individual dynamics. For three coupled CA, we have observed that the evolution of the DA differs qualitatively from the case of two CA but, beyond the critical point, its density vanishes asymptotically. This suggests that stochastic coupling could also be able to synchronize large

populations of CA. A detailed account of these results will be given in a forthcoming work [10].

We have moreover studied the effects of stochastic coupling on elementary CA governed by other evolution rules. It is found that CA whose asymptotic evolution is periodic in time can be synchronized at any value of the coupling probability, and a synchronization transition is observed for complex evolution rules only. In fact, it can be easily proven that coupled CA in periodic states, where the corresponding DA is thus also periodic, will necessarily synchronize within a characteristic relaxation time. In aperiodic CA, instead, the DA density grows as time elapses and synchronization is possible for sufficiently strong coupling only. This would make possible establishing a classification criterion for CA, in terms of their characteristic behavior under the effects of stochastic coupling. Such a criterion could be especially relevant for non-elementary and many-dimensional CA, where the problem of classification is still open.

The synchronization transition that we have here studied in coupled cellular automata is qualitatively similar to critical phenomena shown to occur in a wide class of nonequilibrium systems which have recently attracted a great deal of

interest, including contact processes, chemical reactions, damage spreading, self-organized criticality and directed percolation (DP). Some of those critical phenomena are believed to belong to the same universality class [11], namely, the DP class, with a critical exponent $\beta \approx 0.28$. Such as many other models with vanishing-density transitions, the present system does not seem to belong to the same universality class. In fact, our results show a critical exponent definitely higher than that of the DP transition.

Cellular automata have been successfully applied to model wide classes of natural and artificial systems, such as, for instance, biological populations, physico-chemical processes and computational devices [7]. These systems have also been shown to respond in a nontrivial way to coupling mechanisms [2–4]. Thus, besides the interest of studying coupled cellular automata as an instance of complex, spatially extended dynamical objects able to display mutual synchronization, they could provide insight on the rich dynamics of those real systems.

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- [1] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984).
 - [2] N. Khrustova, G. Vesper, A. Mikhailov, and R. Imbühl, *Phys. Rev. Lett.* **75**, 3564 (1995).
 - [3] D. Golomb, D. Hansel, B. Shraiman, and H. Sompolinsky, *Phys. Rev. A* **45**, 3516 (1992).
 - [4] K. Kaneko, *Physica D* **54**, 5 (1991).
 - [5] J. F. Heagy, T. L. Carrol, and L. M. Pecora, *Phys. Rev. E* **50**, 1874 (1994).
 - [6] A. Amengual, E. Hernández-García, R. Montagne, and M. San Miguel, *Phys. Rev. Lett.* **78**, 4379 (1997).
 - [7] S. Wolfram, *Theory and Applications of Cellular Automata* (World Scientific, New York, 1986).
 - [8] H. J. Herrmann, *Cellular Automata in Nonlinear Phenomena in Complex Systems*, edited by A. N. Proto (North-Holland, Amsterdam, 1989), p. 151.
 - [9] S. Wolfram, *Rev. Mod. Phys.* **55**, 601 (1983).
 - [10] L. G. Morelli and D. H. Zanette (unpublished).
 - [11] R. Dickman, in *Nonequilibrium Statistical Mechanics in One Dimension*, edited by V. Privman (Cambridge University Press, Cambridge, 1997), Part I.